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Property (w) and perturbations III ☆

Pietro Aiena [a,*], Maria T. Biondi [b], Fernando Villafañe [b]

[a] Dipartimento di Metodi e Modelli Matematici, Facoltà di Ingegneria, Università di Palermo, Viale delle Scienze, I-90128 Palermo, PA, Italy
[b] Departamento de Matemáticas, Facultad de Ciencias, Universidad UCLA de Barquisimeto, Venezuela

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ABSTRACT

The property (w) is a variant of Weyl's theorem, for a bounded operator T acting on a Banach space. In this note we consider the preservation of property (w) under a finite rank perturbation commuting with T, whenever T is polaroid, or T has analytical core $K(\lambda_0 I - T) = \{0\}$ for some $\lambda_0 \in \mathbb{C}$. The preservation of property (w) is also studied under commuting nilpotent or under injective quasi-nilpotent perturbations. The theory is exemplified in the case of some special classes of operators.

1. Definitions and basic results

In this paper we continue the study of the class of linear bounded operators defined on Banach spaces that verify property (w), a variant of Weyl's theorem introduced by V. Rakočević in [23] and studied in a more recent paper [8]. The preservation of property (w) under certain classes of perturbations has been investigated in [3,4,7]. In this paper we give further results on the preservation of property (w) in some special cases and improve previous results. Moreover, the theory is applied to several classes of operators. We begin by given some preliminary definitions and basic results.

Let X be an infinite-dimensional complex Banach space and denote by $L(X)$ the algebra of all bounded linear operators on X. A bounded operator $T \in L(X)$ is said to be an *upper semi-Fredholm* operators if $\alpha(T) := \dim\ker T < \infty$ and $T(X)$ is closed, while $T \in L(X)$ is said to be *lower semi-Fredholm* if $\beta(T) := \operatorname{codim} T(X) < \infty$. Let $\Phi_+(X)$ and $\Phi_-(X)$ denote the class of all upper semi-Fredholm operators. The *index* of a semi-Fredholm operator is defined as $\operatorname{ind} T := \alpha(T) - \beta(T)$. $T \in L(X)$ is said to be a *Fredholm operator* if $T \in \Phi_+(X) \cap \Phi_-(X)$. The *upper semi-Weyl* operators $W_+(X)$ are defined as the class of upper semi-Fredholm operators having $\operatorname{ind} T \leqslant 0$. The *lower semi-Weyl* operators $W_-(X)$ are defined as the class of lower semi-Fredholm operators having $\operatorname{ind} T \geqslant 0$. The class of *Weyl* operators is defined by

$$W(X) := W_+(X) \cap W_-(X) = \{T \in \Phi(X):\ \operatorname{ind} T = 0\}.$$

These classes of operators generate the following spectra: the *Weyl spectrum* defined by

$$\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C}:\ \lambda I - T \notin W(X)\},$$

the *upper semi-Weyl spectrum* (in literature called also *Weyl essential approximate point spectrum*) defined by

$$\sigma_{\mathrm{uw}}(T) := \{\lambda \in \mathbb{C}:\ \lambda I - T \notin W_+(X)\},$$

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* Corresponding author.
E-mail addresses: paiena@unipa.it (P. Aiena), mtbiondi@hotmail.com (M.T. Biondi), fvillafa@uicm.ucla.edu.ve (F. Villafañe).

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and the *lower semi-Weyl spectrum* (in literature called also *Weyl essential surjectivity spectrum*) defined by

$$\sigma_{\mathrm{lw}}(T) := \{\lambda \in \mathbb{C}\colon \lambda I - T \notin W_-(X)\}.$$

From the classical Fredholm theory it is known that the classes $W_+(X)$, $W_-(X)$ and $W(X)$ are stable under compact perturbations (also non-commuting perturbations), so that if $K \in L(X)$ is compact then

$$\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{w}}(T+K), \qquad \sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{uw}}(T+K), \qquad \sigma_{\mathrm{lw}}(T) = \sigma_{\mathrm{lw}}(T+K). \tag{1}$$

The *approximate point spectrum* is canonically defined by

$$\sigma_{\mathrm{a}}(T) := \{\lambda \in \mathbb{C}\colon \lambda I - T \text{ is not bounded below}\},$$

where an operator is said to be bounded below if it is injective and has closed range. The approximate point spectrum in general is not stable under finite-rank perturbation, also commuting with T. In fact, the isolated points of $\sigma_{\mathrm{a}}(T)$ and $\sigma_{\mathrm{a}}(T+K)$ can be different. However, a perturbation result holds for accumulation points:

Theorem 1.1. *Suppose that $T \in L(X)$ and let $K \in L(X)$ be a finite rank operator such that $TK = KT$. Then*

(i) $\sigma_{\mathrm{a}}(T+K) \subseteq \sigma_{\mathrm{a}}(T) + \sigma_{\mathrm{a}}(K)$.
(ii) $\operatorname{acc}\sigma_{\mathrm{a}}(T) = \operatorname{acc}\sigma_{\mathrm{a}}(T+K)$, *where* $\operatorname{acc}\sigma_{\mathrm{a}}(T)$ *is the set of all accumulation points of* $\sigma_{\mathrm{a}}(T)$.

Proof. (i) See [21, p. 256]. The equality (ii) has been shown in Theorem 3.2 of [13]. □

For an operator $T \in L(X)$ the *ascent* is defined as $p := p(T) = \inf\{n \in \mathbf{N}\colon \ker T^n = \ker T^{n+1}\}$, while the *descent* is defined as let $q := q(T) = \inf\{n \in \mathbf{N}\colon T^n(X) = T^{n+1}(X)\}$, the infimum over the empty set is taken ∞. It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$ (see [19, Proposition 38.3]). Moreover, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ precisely when λ is a pole of the resolvent of T, see Proposition 50.2 of Heuser [19]. The class of all *upper semi-Browder operators* is defined $B_+(X) := \{T \in \Phi_+(X)\colon p(T) < \infty\}$, while the class of all *lower semi-Browder operators* is defined $B_-(X) := \{T \in \Phi_-(X)\colon q(T) < \infty\}$. The class of all *Browder operators* is defined $B(X) := B_+(X) \cap B_-(X) = \{T \in \Phi(X)\colon p(T) = q(T) < \infty\}$. We have

$$B(X) \subseteq W(X), \qquad B_+(X) \subseteq W_+(X), \qquad B_-(X) \subseteq W_-(X),$$

see [1, Theorem 3.4].

The *Browder spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\mathrm{b}}(T) := \{\lambda \in \mathbb{C}\colon \lambda I - T \notin B(X)\},$$

the *upper semi-Browder spectrum* is defined by

$$\sigma_{\mathrm{ub}}(T) := \{\lambda \in \mathbb{C}\colon \lambda I - T \notin B_+(X)\}.$$

Let X be a complex Banach space and $T \in L(X)$. The operator T is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc $\mathbb{D}$ centered at λ_0, the only analytic function $f : \mathbb{D} \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in \mathbb{D}$ is the function $f \equiv 0$.

An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

Evidently, every operator T, as well as its dual T^*, has SVEP at every point in the boundary of the spectrum $\sigma(T)$, in particular at every isolated point of $\sigma(T)$.

We have the following implications:

$$p(\lambda I - T) < \infty \quad \Rightarrow \quad T \text{ has SVEP at } \lambda, \tag{2}$$

and, dually,

$$q(\lambda I - T) < \infty \quad \Rightarrow \quad T^* \text{ has SVEP at } \lambda, \tag{3}$$

see [1, Theorem 3.8]. Furthermore, from definition of SVEP we have

$$\sigma_{\mathrm{a}}(T) \text{ does not cluster at } \lambda \quad \Rightarrow \quad T \text{ has SVEP at } \lambda. \tag{4}$$

In particular, if the point spectrum $\sigma_{\mathrm{p}}(T)$ ($=$ the set of all eigenvalues of T) is empty then T satisfies SVEP. An important subspace in local spectral theory is the *quasi-nilpotent part* of T defined by

$$H_0(T) := \Big\{x \in X\colon \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\Big\}.$$

We have

$$H_0(\lambda I - T) \text{ closed} \quad \Rightarrow \quad T \text{ has SVEP at } \lambda. \tag{5}$$

Remark 1.2. All the implications (2)–(5) are actually equivalences if we assume that $\lambda I - T$ is semi-Fredholm, see [1, Chapter 3].

2. Property (w)

If $T \in L(X)$, define $p_{00}(T) := \sigma(T) \setminus \sigma_{\mathrm{b}}(T)$ and $p_{00}^a(T) := \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T)$. If $\lambda \in p_{00}^a(T)$ then $p(\lambda I - T) < \infty$, and, since $\lambda I - T$ is upper semi-Fredholm from Remark 1.2 it then follows that $\lambda \in \operatorname{iso} \sigma_{\mathrm{a}}(T)$, so $p_{00}^a(T) \subseteq \pi_{00}^a(T)$, where we set

$$\pi_{00}^a(T) := \{\lambda \in \operatorname{iso} \sigma_{\mathrm{a}}(T):\ 0 < \alpha(\lambda I - T) < \infty\}.$$

Define

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso} \sigma(T):\ 0 < \alpha(\lambda I - T) < \infty\}.$$

Following Harte and W.Y. Lee [18] we say that $T \in L(X)$ satisfies *Browder's theorem* if $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$, while T satisfies *a-Browder's theorem* if $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T)$. Following Coburn [11], we say that *Weyl's theorem holds* for $T \in L(X)$ if $\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T)$.

In a sense Browder's theorem corresponds to half Weyl's theorem:

Theorem 2.1. *(See [2].) If $T \in L(X)$ then Weyl's theorem for T holds precisely when T satisfies Browder's theorem and $\pi_{00}(T) = p_{00}(T)$.*

The following two variants of Weyl's theorem has been introduced by Rakočević [23,24].

Definition 2.2. A bounded operator $T \in L(X)$ is said to satisfy property (w) if

$$\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T),$$

while $T \in L(X)$ is said to satisfy a-Weyl's theorem if

$$\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}^a(T).$$

The relationship between property (w) and a-Browder's theorem is established in the following theorem.

Theorem 2.3. *(See [8].) If $T \in L(X)$ the following statements are equivalent*:

(i) *T satisfies property (w)*;
(ii) *a-Browder's theorem holds for T and $p_{00}^a(T) = \pi_{00}(T)$.*

The following diagram resume the relationships between Weyl's theorems, a-Browder's theorem and property (w).

$$\begin{array}{ccc} \text{Property } (w) & \Rightarrow & a\text{-Browder's theorem} \\ \Downarrow & & \Uparrow \\ \text{Weyl's theorem} & \Leftarrow & a\text{-Weyl's theorem} \end{array}$$

(see [23] and [8]). Examples of operators satisfying Weyl's theorem but not property (w) may be found in [8]. Property (w) is not intermediate between Weyl's theorem and a-Weyl's theorem, see [8] for examples.

Property (w) is fulfilled by a certain number of Hilbert space operators, see [8], and property (w) for T is equivalent to Weyl's theorem for T or to a-Weyl's theorem whenever T^* satisfies SVEP [8, Theorem 2.16]. In particular, property (w) is satisfied by generalized scalar operator T, or if the Hilbert adjoint T' has property $H(p)$ [8, Corollary 2.20].

If $T \in L(X)$, the *analytic core* $K(T)$ is the set of all $x \in X$ such that there exists a constant $c > 0$ and a sequence of elements $x_n \in X$ such that $x_0 = x$, $Tx_n = x_{n-1}$, and $\|x_n\| \leqslant c^n \|x\|$ for all $n \in \mathbb{N}$, see [1] for informations on $K(T)$.

Lemma 2.4. *Suppose that for a bounded operator $T \in L(X)$ there exists $\lambda_0 \in \mathbb{C}$ such that $K(\lambda_0 I - T) = \{0\}$ and $\ker(\lambda_0 I - T) = \{0\}$. Then $\sigma_p(T) = \emptyset$.*

Proof. For all complex $\lambda \neq \lambda_0$ we have $\ker(\lambda I - T) \subseteq K(\lambda_0 I - T)$, so that $\ker(\lambda I - T) = \{0\}$ for all $\lambda \in \mathbb{C}$. □

Let $\mathcal{H}(\sigma(T))$ be the set of all analytic functions defined on a neighborhood of $\sigma(T)$, and for every $f \in \mathcal{H}(\sigma(T))$ let $f(T)$ be defined by means of the classical functional calculus.

Theorem 2.5. *Let $T \in L(X)$ be such that there exists $\lambda_0 \in \mathbb{C}$ such that*

$$K(\lambda_0 I - T) = \{0\} \quad \text{and} \quad \ker(\lambda_0 I - T) = \{0\}. \tag{6}$$

Then property (w) holds for $f(T)$ for all $f \in \mathcal{H}(\sigma(f(T)))$.

Proof. We know from Lemma 2.4 that $\sigma_p(T) = \emptyset$, so T has SVEP. We show that also $\sigma_p(f(T)) = \emptyset$. Let $\mu \in \sigma(f(T))$ and write $\mu - f(\lambda) = p(\lambda)g(\lambda)$, where g is analytic on an open neighborhood $\mathcal{U}$ containing $\sigma(T)$ and without zeros in $\sigma(T)$, p a polynomial of the form $p(\lambda) = \prod_{k=1}^{n}(\lambda_k - \lambda)^{\nu_k}$, with distinct roots $\lambda_1, \dots, \lambda_n$ lying in $\sigma(T)$. Then

$$\mu I - f(T) = \prod_{k=1}^{n}(\lambda_k I - T)^{\nu_k} g(T).$$

Since $g(T)$ is invertible, $\sigma_{\mathrm{p}}(T) = \emptyset$ implies that $\ker(\mu I - f(T)) = \{0\}$ for all $\mu \in \mathbb{C}$, so $\sigma_p(f(T)) = \emptyset$. Since T has SVEP then $f(T)$ has SVEP, see Theorem 2.40 of [1], so that a-Browder's theorem holds for $f(T)$ [5]. To prove that property (w) holds for $f(T)$, by Theorem 2.3 it then suffices to prove that

$$p_{00}^{a}\big(f(T)\big) = \pi_{00}\big(f(T)\big).$$

Obviously, the condition $\sigma_p(f(T)) = \emptyset$ entails that

$$\pi_{00}\big(f(T)\big) = \pi_{00}^{a}\big(f(T)\big) = \emptyset.$$

On the other hand, the inclusion $p_{00}^{a}(f(T)) \subseteq \pi_{00}^{a}(f(T))$ holds for every operator $T \in L(X)$, so also $p_{00}^{a}(f(T))$ is empty. By Theorem 2.3 it then follows that $f(T)$ satisfies property (w). □

The conditions of Theorem 2.5 are satisfied by any injective operator for which the *hyperrange* $T^{\infty}(X) := \bigcap T^n(X)$ is $\{0\}$. In fact, $K(T) \subseteq T^{\infty}(X)$ for all $T \in L(X)$, so that $K(T) = \{0\}$. In particular, the conditions of Theorem 2.5 are satisfied by a *semi-shift* T, i.e. T is an isometry for which $T^{\infty}(X) = \{0\}$, see [21] for details on this class of operators. Clearly, a semi-shift T on a non-trivial Banach space is a non-invertible isometry.

Theorem 2.6. *If $T \in L(X)$ is a semi-shift and $K \in L(X)$ is a finite-rank operator which commutes with T then property (w) holds for $f(T) + K$ for all $f \in \mathcal{H}(\sigma(f(T)))$.*

Proof. Since T is a non-invertible isometry, the approximate point spectrum $\sigma_{\mathrm{a}}(T)$ is the closed unit circle of $\mathbb{C}$, see [21, Proposition 1.6.2]. Hence $\operatorname{iso}\sigma_{\mathrm{a}}(T) = \emptyset$ and the result then follows from Theorems 2.5 and 2.7 of [3]. □

Let $\mathcal{P}_0(X)$ denote the class of all operators for which there exists $p := p(\lambda) \in \mathbb{N}$ such that

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \text{for all } \lambda \in \pi_{00}(T).$$

Of course we assume that $T \in P_0(X)$ whenever $\pi_{00}(T)$ is empty. By Theorem 2.2 of [6] we have that $T \in \mathcal{P}_0(X)$ if and only if $\pi_{00}(T) = p_{00}(T)$, where $p_{00}(T) := \sigma(T) \setminus \sigma_{\mathrm{b}}(T)$. The next theorem shows that the condition $T \in \mathcal{P}_0(X)$ entails Weyl's theorem whenever either T or T^* have SVEP.

Theorem 2.7. *Suppose that T has SVEP at the points $\lambda \notin \sigma_{\mathrm{w}}(T)$, or T^* has SVEP at the points $\lambda \notin \sigma_{\mathrm{lw}}(T)$. If $T \in \mathcal{P}_0(X)$ then Weyl's theorem holds for T.*

Proof. As observed before, the condition $T \in \mathcal{P}_0(X)$ is equivalent saying that $\pi_{00}(T) = p_{00}(T)$, while either the conditions that T has SVEP at the points $\lambda \notin \sigma_{\mathrm{w}}(T)$, or that T^* has SVEP at the points $\lambda \notin \sigma_{\mathrm{lw}}(T)$ entail Browder's theorem holds for T, see [5, Theorem 2.3]. □

The following examples show that the conditions of Theorem 2.7 do not imply that T satisfies property (w).

Example 2.8. Let $X := \ell^2(\mathbb{N})$ and $R \in L(X)$ be the unilateral right shift. Define

$$U(x_1, x_2, \dots) := (0, x_2, x_3, \dots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$

Observe that U is an orthogonal projection, so is self-adoint. If $T := R \oplus U$ then $\sigma(T) = \mathbf{D}$, $\mathbf{D}$ the closed unit disc, so that $\operatorname{iso}\sigma(T) = p_{00}(T) = \pi_{00}(T) = \emptyset$ and hence $T \in \mathcal{P}_0(X)$. It is easy to check that $\sigma_{\mathrm{w}}(T) = \mathbb{D}$.

On the other hand we have $\sigma_{\mathrm{a}}(T) = \Gamma \cup \{0\}$, Γ the unit circle of $\mathbb{C}$. By (4) T has SVEP at 0 and at all points $\lambda \notin \sigma_{\mathrm{a}}(T)$. Since T has SVEP at every point in the boundary of the spectrum it then follows that T has SVEP. This implies that a-Browder's theorem holds for T [5], i.e. $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T)$. It is easily seen that $\sigma_{\mathrm{uw}}(T) = \Gamma$, so $0 \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{ub}(T) = p_{00}^{a}(T)$.

Therefore, $p_{00}^a(T) \neq \pi_{00}(T)$ and hence, by Theorem 2.3, T does not satisfy property (w). Note that by Theorem 2.7 T satisfies Weyl's theorem.

Analogously, let $L \in L(X)$ denote the unilateral left shift, and consider $S := L \oplus U$. Since $L' = R$ and U is self-adjoint, then $S' = L' \oplus U' = R \oplus U = T$ has SVEP, by the first part, or equivalently the dual S^* has SVEP, see [2]. Clearly, $S \in \mathcal{P}_0(X)$, since $\sigma(S) = \overline{\sigma(S')} = \mathbf{D}$. From the equalities $\sigma_{ub}(S) = \overline{\sigma_{ub}(S')} = \overline{\sigma_{ub}(T)}$ and $\sigma_a(S) = \overline{\sigma_a(S')} = \overline{\sigma_a(T)}$, we deduce that $p_{00}^a(S) = \overline{p_{00}^a(T)}$. From Theorem 2.7 we know that S satisfies Weyl's theorem, so

$$\pi_{00}(S) = \sigma(S) \setminus \sigma_w(S) = \overline{\sigma(T) \setminus \sigma_w(T)} = \overline{\pi_{00}(T)}.$$

Therefore, $\pi_{00}(S) \neq p_{00}^a(S)$, consequently S does not satisfy property (w).

A bounded operator $T \in L(X)$ is said to be *polaroid* (respectively, *a-polaroid*) if $\operatorname{iso}\sigma(T) = \emptyset$ or every isolated point of $\sigma(T)$ is a pole of the resolvent of T (respectively, if $\operatorname{iso}\sigma_a(T) = \emptyset$ or every isolated point of $\sigma_a(T)$ is a pole of the resolvent of T).

Let $\mathcal{P}(X)$ denote the class of all operators $T \in L(X)$ such that $\operatorname{iso}\sigma(T) = \emptyset$, or there exists $p := p(\lambda) \in \mathbb{N}$ such that

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \text{for all } \lambda \in \operatorname{iso}\sigma(T).$$

Evidently, $\mathcal{P}(X) \subseteq \mathcal{P}_0(X)$.

Theorem 2.9. *If $T \in \mathcal{P}(X)$ if and only if T is polaroid. Moreover, if $T \in \mathcal{P}(X)$, we have*:

(i) *If T^* has SVEP then property (w) holds for T.*
(ii) *If T has SVEP then property (w) holds for T^*.*
(iii) *If both T and T^* have SVEP then property (w) holds for T and T^*.*

Proof. Suppose $T \in \mathcal{P}(X)$ and that λ is isolated point of $\sigma(T)$. Then there exists $p \in \mathbb{N}$ such that $H_0(\lambda I - T) = \ker(\lambda I - T)^p$. Since λ is isolated in $\sigma(T)$ then, by [1, Theorem 3.74],

$$X = H_0(\lambda I - T) \oplus K(\lambda I - T) = \ker(\lambda I - T)^p \oplus K(\lambda I - T),$$

from which we obtain

$$(\lambda I - T)^p(X) = (\lambda I - T)^p\big(K(\lambda I - T)\big) = K(\lambda I - T),$$

so

$$X = \ker(\lambda I - T)^p \oplus (\lambda I - T)^p(X),$$

which implies, by [1, Theorem 3.6], that $p(\lambda I - T) = q(\lambda I - T) \leqslant p$, hence λ is a pole of the resolvent, so that T is polaroid.

Conversely, suppose that T is polaroid and λ is an isolated point of $\sigma(T)$. Then λ is a pole, and if p is its order then $H_0(\lambda I - T) = \ker(\lambda I - T)^p$, see Theorem 3.74 of [1].

The assertions (i)–(iii) have been shown in [7, Theorem 2.24]. $\square$

A bounded operator $T \in L(X)$ on a Banach space X is said to be *paranormal* if

$$\|Tx\|^2 \leqslant \|T^2x\|\,\|x\| \quad \text{holds for all } x \in X.$$

An operator $T \in L(X)$ for which there exists a complex nonconstant polynomial h such that $h(T)$ is paranormal is said to be *algebraically paranormal*. The class $P(X)$ is rather large. In fact, every algebraic paranormal operator defined on a Hilbert space is polaroid, see [2]. Clearly, $P(X)$ contains the class of operators that satisfy the following property $H(p)$:

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \text{for all } \lambda \in \mathbb{C}.$$

The class $H(p)$ has been introduced in [22] and in [9] this class of operators has been studied for $p := p(\lambda) = 1$ for all $\lambda \in \mathbb{C}$. Property $H(p)$ is satisfied by every generalized scalar operator, and in particular for p-hyponormal, log-hyponormal, M-hyponormal operators on Hilbert spaces, see [22]. In the case of Hilbert space operators the condition T^* has SVEP may be replaced by the condition that the Hilbert adjoint T' has SVEP, see [2]. Property $H(1)$ is satisfied by multipliers of commutative semi-simple Banach algebras, in particular by convolution operators on group algebras [9]. Every paranormal operator on a Hilbert space satisfies SVEP [6]. Consequently, by Theorem 2.40 of [1], every algebraically paranormal satisfies SVEP. The SVEP is also satisfied by $H(p)$ operators, since by (5) the condition that $H_0(\lambda I - T)$ is closed entails that T has SVEP at λ. Another class of polaroid operators, different from the class $H(p)$ and which contains properly the class of paranormal operators, is the class CHN of *completely hereditarily operators*, see [14]. Here $T \in L(X)$ is said to be *normaloid* if $\|T\|$ is equal to the spectral radius of T, while $T \in$ CHN if either every part (= restriction of T on a closed invariant subspace), and also every invertible part is normaloid, or $\lambda I - T$ is normaloid for all $\lambda \in \mathbb{C}$.

3. Property (w) under perturbations

In this section we shall give some conditions for which property (w) is preserved under commuting finite-rank or quasi-nilpotent perturbations. Recall that a bounded operator $T \in L(X)$ is said to be *isoloid* (respectively, *a-isoloid*) if every isolated point of $\sigma(T)$ (respectively, every isolated point of $\sigma_a(T)$) is an eigenvalue of T. Every a-isoloid operator is isoloid. This is easily seen: if T is a-isoloid and $\lambda \in \operatorname{iso}\sigma(T)$ then $\lambda \notin \sigma_a(T)$ or $\lambda \in \sigma_a(T)$. In the first case $\lambda I - T$ is bounded below, in particular upper semi-Fredholm. The SVEP of both T and T^* at λ then implies that $p(\lambda I - T) = q(\lambda I - T) < \infty$, so λ is a pole. Obviously, also in the second case λ is a pole, since by assumption T is a-isoloid.

As a-Weyl's theorem, property (w) is not preserved under finite rank perturbations (also commuting finite rank perturbations).

Example 3.1. Let $T := Q \oplus I$ defined on $X \oplus X$, where Q is an injective quasi-nilpotent operator. It is easily seen that T satisfies a-Weyl's theorem. Define $K := 0 \oplus (-P)$, where P is a finite rank projection. Then $TK = KT$, and since T^* has a finite spectrum then T^* has SVEP, hence $T^* + K^*$ has SVEP, by Lemma 2.8 of [3]. Therefore $\sigma(T+K) = \sigma_a(T+K)$, by Corollary 2.45 of [1]. On the other hand it is easy to see that $0 \in \sigma(T+K) \cap \sigma_{uw}(T+K)$, so $0 \notin \sigma_a(T+K) \setminus \sigma_{uw}(T+K)$, while $0 \in \pi_{00}^a(T+K) = \pi_{00}(T+K)$, thus $T+K$ does not verify property (w).

The following result has been shown in [13].

Theorem 3.2. *If $T \in L(X)$ is a-isoloid and satisfies a-Weyl's theorem then $T+K$ satisfies a-Weyl's theorem for every finite-dimensional operator $K \in L(X)$ commuting with T.*

An analogous result of that of Theorem 3.2 does not hold for property (w), see [3] for a counter-example. However, property (w) is preserved if to the assumptions of Theorem 3.2 we add the assumption that $\sigma_a(T) = \sigma_a(T+K)$ [4].

The result of Theorem 2.9 may be improved as follows:

Theorem 3.3. *Suppose that $T \in P(X)$ and K is a finite rank operator commuting with T.*

(i) *If T^* has SVEP then $f(T) + K$ satisfies property (w) for all $f \in \mathcal{H}(\sigma(T))$.*
(ii) *If T has SVEP then $f(T^*) + K^*$ satisfies property (w) for all $f \in \mathcal{H}(\sigma(T))$.*

Proof. (i) We know that T is polaroid. By [1, Corollary 2.45] we have $\sigma_a(T) = \sigma(T)$, so T is a-polaroid and hence a-isoloid. By Theorem 2.22 of [8] it then follows that $f(T)$ has property (w) for all $f \in \mathcal{H}(\sigma(T))$. Now, by Theorem 2.40 of [1] $f(T^*) = f(T)^*$ has SVEP, so that, by Theorem 2.16 of [8] a-Weyl's theorem holds for $f(T)$. Since $f(T)$ and K commutes, by Theorem 3.2 we then obtain that $f(T) + K$ satisfies a-Weyl's theorem. By Lemma 2.8 of [3] $f(T)^* + K^* = (f(T) + K)^*$ has SVEP. This implies that property (w) and a-Weyl's theorem for $f(T) + K$ are equivalent, again by Theorem 2.16 of [8], so the proof is complete.

(ii) The argument is analogous to that of part (i). Just observe that $\sigma_a(T^*) = \sigma(T^*)$ by [1, Corollary 2.45], so that T^* is a-polaroid, hence a-isoloid. Moreover, by Theorem 2.22 of [8] it then follows that $f(T^*)$ has property (w) for all $f \in \mathcal{H}(\sigma(T))$. By Theorem 2.40 of [1] $f(T) = f(T)$ has SVEP, so that, so, by Theorem 2.16 of [8] a-Weyl's theorem holds for $f(T^*)$. Since $f(T^*)$ and K^* commutes, by Theorem 3.2 we then obtain that $f(T^*) + K^*$ satisfies a-Weyl's theorem. Again by Lemma 2.8 of [3] $f(T) + K$ has SVEP, so that (w) and a-Weyl's theorem for $f(T^*) + K^*$ are equivalent, by Theorem 2.16 of [8]. $\square$

The basic role of SVEP arises in local spectral theory since for all decomposable operators both T and T^* have SVEP. Every *generalized scalar* operator on a Banach space is decomposable (see [21] for relevant definitions and results). In particular, every *spectral operators of finite type* is decomposable [12, Theorem 3.6].

Corollary 3.4. *Suppose that $T \in L(X)$ is generalized scalar and K is a finite rank operator commuting with T. Then property (w) holds for both $f(T) + K$ and $f(T^*) + K^*$. In particular, this is true for every spectral operator of finite type.*

Proof. Both T and T^* have SVEP. Moreover, every generalized scalar operator T has property $H(p)$ [22, Example 3], so T is polaroid. The second statement is clear: every spectral operators of finite type is generalized scalar. $\square$

As observed before, in the case of Hilbert space operators, the condition that T^* has SVEP in part (i) of Theorem 3.3 may be replaced by the condition that the Hilbert adjoint T' has SVEP, since the SVEP for T' and T^* are equivalent. Moreover, property (w) for the Banach space dual T^* is equivalent to property (w) for T', see [7].

Theorem 3.3, applies to several classes of operators. For instance, if $T \in L(H)$, H a Hilbert space, is algebraically paranormal, or $T \in H(p)$. In this case T is polaroid and T has SVEP, so part (ii) of Theorem 3.2 applies.

A subclass of paranormal operators on Hilbert spaces, is given by the class of A operators introduced by Furuta, Ito and Yamazaki [16]. $T \in L(H)$ is said to belong to the class A if $|T^2| \geqslant |T|^2$. T is said to be an *analytically class A* operator if there exists some $f \in \mathcal{H}(\sigma(T))$ such that $f(T)$ belongs to the class A. A class A operators satisfies SVEP, since paranormal.

Corollary 3.5. *If $T' \in L(H)$ is an analytically class A operator and $K \in L(H)$ is a finite rank operator which commutes with T then property (w), or equivalently, a-Weyl's theorem holds for $f(T) + K$ for all $f \in \mathcal{H}(\sigma(T))$.*

Proof. Assume that T' is an analytically class A operator. Then $f(T')$ is a class A operator, so $f(T')$ has SVEP and hence also T' has SVEP, see [1, Theorem 2.40]. Moreover, T' is polaroid [10, Lemma 3.3], and hence by [7, Theorem 2.5] also T is polaroid. Therefore, by Theorem 3.3, $f(T) + K$ satisfies property (w). Since $f(T') + K' = (f(T) + K)'$ has SVEP, see Lemma 2.8 of [3] property (w) and a-Weyl's theorem for $f(T) + K$ are equivalent. □

It should be noted that the result of Corollary 3.5 extends Theorem 3.6 of [10].

An operator $T \in$ CHN in general does not satisfy SVEP. Recall that a subspace M of a Banach space X is said to be *orthogonal* (in the sense of Birkhoff) to an other subspace N of X, $M \perp N$ if

$$\|x\| \leqslant \|x + y\| \quad \text{for all } x \in M, \ y \in N.$$

It should be noted that this asymmetric definition coincides with the usual concept of orthogonality in the case that X is a Hilbert space. M and N are said to be *mutually orthogonal*, $M \perp_m N$ if $M \perp N$ and $N \perp M$.

$$\ker(\mu I - T) \perp_m \ker(\lambda I - T) \quad \text{for all } \lambda \neq \mu, \ \lambda \neq 0. \tag{7}$$

The condition (7) entails SVEP for T. In fact, suppose that U is an open disc and $f : U \to X$ an analytic function such that $0 \neq f(z) \in \ker(zI - T)$ for all $z \in U$. Then f fails to be continuous at every $0 \neq \lambda \in U$. Hence f is identically 0 on U, i.e. T has SVEP. Note that the condition (7) is satisfied by paranormal operators. Recalling that every $T \in$ CHN is polaroid, by Theorem 3.3, part (i) we then have:

Corollary 3.6. *If $T \in$ CHN satisfies the orthogonality condition* (7)*, K is a finite-rank operator which commutes with T, then $f(T^*) + K^*$ satisfies property (w) for all $f \in \mathcal{H}(\sigma(T))$.*

A Hilbert space operator $T \in L(H)$ is said (p,k)-*quasihyponormal* if

$$T'^k\big(|T|^{2p} - |T'|^{2p}\big)T^k \geqslant 0$$

for some $0 < p \leqslant 1$ [20]. This class of operators does not fit into either of the classes $H(p)$ and CHN. By [26, Theorem 6] every (p,k)-quasihyponormal is polaroid, and by [15] we have $p(\lambda I - T) < \infty$ for all $\lambda \in \mathbb{C}$, so these operators have SVEP.

Corollary 3.7. *If $T \in L(H)$ (p,k)-quasihyponormal, $K \in L(H)$ is a finite-rank operator which commutes with T, then $f(T') + K'$ satisfies property (w) for all $f \in \mathcal{H}(\sigma(T))$.*

The next results deal with quasi-nilpotent perturbations. We first recall two well-known results: if Q a quasi-nilpotent operator commuting with $T \in L(X)$, then

$$\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(T + Q) \quad \text{and} \quad \sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{uw}}(T + Q). \tag{8}$$

Since $\sigma(T + Q) = \sigma(T)$ and $\sigma_{\mathrm{b}}(T + Q) = \sigma_{\mathrm{b}}(T)$ (for the last equality see [25]), we then have $p_{00}(T + Q) = p_{00}(T)$.

It is easily seen that property (w) is transmitted under commuting nilpotent perturbations N.

Theorem 3.8. *If $T \in L(X)$ satisfies property (w), $N \in L(X)$ is a nilpotent operator commuting with T then $T + N$ satisfies property (w).*

Proof. If T satisfies property (w) then T satisfies Weyl's theorem, so by Theorem 2.1, $\pi_{00}(T) = p_{00}(T)$, or equivalently $T \in \mathcal{P}_0(X)$. This implies that $T + N \in \mathcal{P}_0(X)$, see Lemma 2.7 of [6], and hence

$$\pi_{00}(T + N) = p_{00}(T + N) = p_{00}(T) = \pi_{00}(T).$$

From (8) we know that $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(T + N)$ and $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{uw}}(T + N)$, hence

$$\sigma_{\mathrm{a}}(T + N) \setminus \sigma_{\mathrm{uw}}(T + N) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T) = \pi_{00}(T + N),$$

i.e. $T + N$ satisfies property (w). □

The result of Theorem 3.8 improves Theorem 2.8 of [4], where the preservation of property (w), under a commuting nilpotent perturbation, was proved under the assumption that T is a-isoloid. Generally, property (w) is not transmitted from T to a quasi-nilpotent perturbation $T+Q$. In fact, if $Q \in L(\ell^2(\mathbb{N}))$ is defined by

$$Q(x_1, x_2, \ldots) = \left(\frac{x_2}{2}, \frac{x_3}{3}, \ldots\right) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$

Then Q is quasi-nilpotent and

$$\{0\} = \pi_{00}(Q) \neq \sigma_{\mathrm{a}}(Q) \setminus \sigma_{\mathrm{uw}}(Q) = \emptyset.$$

Take $T = 0$. Clearly, T satisfies property (w) but $T+Q = Q$ fails this property. Note that Q is not injective. The following result has been proved in [4]. We shall give a very simpler proof.

Lemma 3.9. *Let $T \in L(X)$ be such that $\alpha(T) < \infty$. Suppose that there exists an injective quasi-nilpotent operator $Q \in L(X)$ such that $TQ = QT$. Then T is injective.*

Proof. Set $Y := \ker T$. Clearly, Y is invariant under Q and the restriction $(\lambda I - Q)|Y$ is injective for all $\lambda \neq 0$. Since Y is finite-dimensional then $(\lambda I - Q)|Y$ is also surjective for all $\lambda \neq 0$, thus $\sigma(Q|Y) \subseteq \{0\}$. On the other hand, from assumption we know that $Q|Y$ is injective and hence $Q|Y$ is surjective, so $\sigma(Q|Y) = \emptyset$, from which we conclude that $Y = \{0\}$. □

Theorem 3.10. *Suppose that for $T \in L(X)$ there exists an injective quasi-nilpotent Q operator commuting with T. Then both T and $T+Q$ satisfy property (w), a-Weyl's and Weyl's theorem.*

Proof. We show first property (w) for T. It is evident, by Lemma 3.9, that $\pi_{00}(T)$ is empty.

Suppose that $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T)$ is not empty and let $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T)$. Since $\lambda I - T \in W_+(X)$ then $\alpha(\lambda I - T) < \infty$ and $\lambda I - T$ has closed range. Since $\lambda I - T$ commutes with Q it then follows, by Lemma 3.9, that $\lambda I - T$ is injective, so $\lambda \notin \sigma_{\mathrm{a}}(T)$, a contradiction. Therefore, also $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T)$ is empty. Therefore, property (w) holds for T.

To show that a-Weyl's theorem holds for T observe that by Lemma 3.9, also $\pi_{00}^a(T)$ is empty, hence

$$\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}^a(T) = \emptyset.$$

Analogously, a-Weyl's theorem also holds for $T+Q$, since the operator $T+Q$ commutes with Q.

Weyl's theorem is obvious: property (w), as well as a-Weyl's theorem, entails Weyl's theorem. Property (w), as well as a-Weyl's theorem and Weyl's theorem, for $T+Q$ is clear, since also $T+Q$ commutes with Q. □

Theorem 3.10 shows that in [17, Theorem 2.4] the assumption that T satisfies Weyl's theorem is redundant. Analogously, the condition that T satisfies property (w) in [4, Theorem 2.13] is redundant. Obviously, by Theorem 3.10 an injective quasi-nilpotent operator satisfies property (w).

Example 3.11. In Theorem 3.10 the condition *quasi-nilpotent* cannot be replaced by the condition *compact*. For example consider the following operators $T := U \oplus I$ and $K := V \oplus Q$ on $\ell_2(\mathbb{N}) \oplus \ell_2(\mathbb{N})$, where, $Q \in L(\ell_2(\mathbb{N}))$ is an injective compact quasi-nilpotent operator,

$$Ux := \left(0, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\right), \quad x := (x_n)_{n=1,2,\ldots} \in \ell_2(\mathbb{N}),$$

and

$$Vx := \left(1, -\frac{x_2}{2}, -\frac{x_3}{3}, \ldots\right), \quad x := (x_n)_{n=1,2,\ldots} \in \ell_2(\mathbb{N}).$$

The operator U is compact, so $T = U \oplus I$ and T^* have SVEP since both operator have discrete spectrum. Consequently, by [3, Theorem 1.5] $\sigma_{\mathrm{a}}(T) = \sigma(T)$ and $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{w}}(T) = \{0, 1\}$. Clearly,

$$\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T) = \left\{\frac{1}{n}\colon\ n = 2, 3, \ldots\right\},$$

thus property (w) holds for T. Note that K is an injective compact operator, $TK = KT$ and

$$\sigma(T+K) = \sigma_{\mathrm{w}}(T+K) = \{0, 1\} \quad \text{and} \quad \pi_{00}(T+K) = \{1\}. \tag{9}$$

Note that $T^* + K^*$ has SVEP, since has finite spectrum, again by [3, Theorem 1.5],

$$\sigma(T+K) = \sigma_{\mathrm{a}}(T+K) \quad \text{and} \quad \sigma_{\mathrm{uw}}(T+K) = \sigma_{\mathrm{w}}(T+K).$$

From the equalities (9) we then deduce that property (w) does not holds for $T+K$.

From Lemma 3.9 we deduce that if $0 < \alpha(T) < \infty$ then there exists no injective quasi-nilpotent operator Q which commutes with T. Theorem 3.10 has the following interesting consequence:

Corollary 3.12. *Suppose that T does not satisfy Browder's theorem. Then there exists no injective quasi-nilpotent operator commuting with T.*

Proof. If T does not satisfy Browder's theorem then T does not satisfy Weyl's theorem, so the assertion follows from Theorem 3.10. □

Remark 3.13. It is known that for a finite rank operator K and every $T \in L(X)$ then $\alpha(T) = \infty$ if and only if $\alpha(T + K) = \infty$.

As noted above, a non-injective quasi-nilpotent operator Q may fail property (w).

Theorem 3.14. *Suppose that $Q \in L(X)$ is a quasi-nilpotent and $K \in L(X)$ is a finite rank operator commuting with Q. If Q satisfies property (w) then $Q + K$ satisfies property (w).*

Proof. If Q is injective then $Q + K$ satisfies property (w) by Theorem 3.10. Suppose that Q is non-injective and that satisfies property (w). Clearly, $\{0\} = \sigma_{\mathrm{uw}}(Q) = \sigma_{\mathrm{a}}(Q)$ since both $\sigma_{\mathrm{uw}}(Q)$ and $\sigma_{\mathrm{a}}(Q)$ are non-empty, and from the equalities (8) we know that

$$\{0\} = \sigma_{\mathrm{uw}}(Q) = \sigma_{\mathrm{uw}}(Q + K)$$

and $\sigma_{\mathrm{a}}(Q + K) = \sigma_{\mathrm{a}}(K)$, so that $\sigma_{\mathrm{a}}(Q + K) \setminus \sigma_{\mathrm{uw}}(Q + K)$ is the set of all non-zero eigenvalues of K. Say $\lambda_1, \dots, \lambda_n$.

We show that $\pi_{00}(Q + K) = \{\lambda_1, \dots, \lambda_n\}$. Since Q satisfies property (w) we have

$$\emptyset = \sigma_{\mathrm{a}}(Q) \setminus \sigma_{\mathrm{uw}}(Q) = \pi_{00}(Q),$$

and since $\alpha(Q) > 0$ this implies that $\alpha(Q) = \infty$. As observed in Remark 3.13, this implies that $\alpha(Q + K) = \infty$, so that $0 \notin \pi_{00}(Q + K)$. Therefore,

$$\pi_{00}(Q + K) \subseteq \sigma_{\mathrm{a}}(Q + K) = \sigma_{\mathrm{a}}(K) \setminus \{0\} = \{\lambda_1, \dots, \lambda_n\}.$$

We show the opposite inclusion. For every $i = 1, \dots, n$ the operators $\lambda_i I - Q$ are invertible, in particular Fredholm operators, so that $\lambda_i I - (Q + K)$ is a Fredholm operator. Therefore $\alpha(\lambda_i I - (Q + K)) < \infty$ and $\lambda_i I - (Q + K)$ has closed range.

Now, suppose that $\alpha(\lambda_i I - (Q + K)) = 0$. Then $\lambda_i \notin \sigma_{\mathrm{a}}(Q + K) = \sigma_{\mathrm{a}}(K)$, hence $\lambda_i I - K$ is injective. Since K is a finite-rank operator it then follows that

$$\alpha(\lambda_i I - K = \beta(\lambda_i I - K) = 0,$$

i.e. $\lambda_i \notin \sigma(K)$, a contradiction. Therefore $\lambda_i \in \pi_{00}(Q + K)$, and consequently property (w) holds for $Q + K$. □

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